Limits

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1 Numbers

All numbers in these notes are real numbers. The set of all real numbers is denoted by $\mathbb{R}$.

The most important subsets of real numbers are the set of natural numbers, denoted by $\mathbb{N}$, and the set of integers, denoted by $\mathbb{Z}$. That is

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{Z} = \{-n : n \in \mathbb{N}\} \cup \{0\} \cup \mathbb{N}.$$

Important subsets of $\mathbb{R}$ are intervals. Let a and b be real numbers such that $a < b$. Here are all possible intervals with endpoints a and b:

$$\begin{array}{llll} x \in [a,b] & \text{means} \ \ a \leq x \leq b, & x \in (a,b) & \text{means} \ \ a < x < b, \\ x \in [a,b) & \text{means} \ \ a \leq x < b, & x \in (a,b] & \text{means} \ \ a < x \leq b. \end{array}$$

The set $[a,b]$ is called a *closed interval*. The set (a,b) is called an *open interval*. The sets $[a,b)$ and $(a,b]$ is called *half-open interval* or *half-closed interval*. These intervals are called *finite intervals*. The *infinite intervals* are

$$\begin{array}{ll} [a,+\infty) = \bigl\{x \in \mathbb{R} : x \geq a\bigr\}, & (a,+\infty) = \bigl\{x \in \mathbb{R} : x > a\bigr\}, \\ (-\infty,a] = \bigl\{x \in \mathbb{R} : x \leq a\bigr\}, & (-\infty,a) = \bigl\{x \in \mathbb{R} : x < a\bigr\}. \end{array}$$

Let u and v be real numbers. We define $\max\{u,v\} := \begin{cases} v & \text{if} \quad u \leq v, \\ u & \text{if} \quad v < u. \end{cases}$

Notice the fact that $\max\{u,v\} = \max\{v,u\}$.

2 Functions

2.1 The definition

Next we review the definition of a function. Let A and B be sets. A *function* f from A to B is a rule that assigns **exactly one** element of B to **each** element in A. This relationship between the sets A and B and the rule f is indicated by the following notation: $f : A \to B$. If $x \in A$ the unique element of B which is assigned to x by the function f is called the value of f at x. This element is denoted by $f(x)$. The set A is *domain* of f. The subset $\{f(x) \in B \ : \ x \in A\}$ of B is the *range* of f.

In this class we are interested in functions for which both sets A and B are subsets of the set of real numbers $\mathbb{R}$. Some examples of such functions are given below.

2.2 Examples

For each of the examples below answer the following questions: (a) What are the domain and the range of the function? (b) Plot the function using your graphing calculator. Plot the function by hand emphasizing the details missed by your graphing calculator.

Example 2.2.1. Let $\operatorname{Sign} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{Sign}(x) := \begin{cases} 1 & \text{for } \ x > 0, \\ 0 & \text{for } \ x = 0, \\ -1 & \text{for } \ x < 0. \end{cases}$$

This function is called the *sign* function.

Example 2.2.2. Let $\operatorname{UnitStep} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{UnitStep}(x) := \begin{cases} 1 & \text{for } \ x \geq 0, \\ 0 & \text{for } \ x < 0. \end{cases}$$

This function is called the *unit step* function.

Example 2.2.3. Let $\operatorname{Floor} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{Floor}(x) = \lfloor x \rfloor := \max\bigl\{k \in \mathbb{Z} : k \leq x\bigr\}.$$

This function is called the *floor* function. In other words for a given $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the unique integer with the following property

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

As an immediate consequence we get that

$$x - 1 < \operatorname{Floor}(x) \leq x \qquad \text{for all} \qquad x \in \mathbb{R}.$$

Example 2.2.4. Let $\operatorname{Ceiling} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{Ceiling}(x) = \lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}.$$

This function is called the *ceiling* function.

(a) Prove that $x \leq \operatorname{Ceiling}(x) < x + 1$ for all $x \in \mathbb{R}$.

Example 2.2.5. Let $\operatorname{Abs} : \mathbb{R} \to \mathbb{R}$ be given by the formula

$$\operatorname{Abs}(x) = |x| := \begin{cases} x & \text{if } \ x \geq 0, \\ -x & \text{if } \ x < 0. \end{cases}$$

This function is called the *absolute value* function.

Exercise 2.2.6. Prove that $\max\{u, v\} = v + (u - v)\operatorname{UnitStep}(u - v)$ for all $u, v \in \mathbb{R}$.

2.3 Absolute value

For a given real number a the number $|a|$ is called the *absolute value* of the number a.

From the calculus you are familiar with the geometric representation of real numbers as points on a straight line. This is done by choosing a point on the line to represent 0 and another point to represent 1. Then, every real number will correspond to a point on this line (called the number line), and every point on the number line will correspond to a real number. This geometric representation might be very helpful in doing the problems.

Geometrically, the absolute value of a represents the distance between 0 and a, or, generally $|a-b|$ is the *distance* between the real numbers a and b on the number line.

The basic properties of the absolute value are given in the following exercises.

Exercise 2.3.1. Prove the following statements.

(i) $|x| = \max\{x, -x\}$.

(ii) $|x| \geq 0$ for all $x \in \mathbb{R}$.

(iii) $|-x| = |x|$ for all $x \in \mathbb{R}$.

(iv) $-x \leq |x|$ and $x \leq |x|$ for all $x \in \mathbb{R}$.

(v) $|xy| = |x||y|$ for all $x, y \in \mathbb{R}$.

(vi) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ for all $x, y \in \mathbb{R}$, $y \neq 0$.

Proof. To prove (i) we consider two cases. **Case I.** Assume $x \geq 0$. Then $-x \leq 0$. Since $-x \leq 0$ and $0 \leq x$, it follows that $-x \leq x$. Therefore $\max\{x, -x\} = x$. By Definition in Example 2.2.5 for $x \geq 0$ we have that $\mathrm{Abs}(x) = x$. Hence, we conclude that $\mathrm{Abs}(x) = \max\{x, -x\}$ in this case. **Case II.** Assume $x < 0$. Then $-x > 0$. Since $-x > 0$ and $0 > x$, it follows that $-x > x$. Therefore $\max\{x, -x\} = -x$. By Definition in Example 2.2.5 for $x < 0$ we have that $\mathrm{Abs}(x) = -x$. Hence, we conclude that $\mathrm{Abs}(x) = \max\{x, -x\}$ in this case.

Since Cases I and II include all real numbers x, the equality $\mathrm{Abs}(x) = \max\{x, -x\}$ is proved.

The statement (ii) can also be proved by considering two cases.

To prove (iii) note that by (i) $|x| = \max\{x, -x\}$ and also $|-x| = \max\{-x, -(-x)\} = \max\{-x, x\}$. Since $\max\{x, -x\} = \max\{-x, x\}$, we conclude that $|x| = |-x|$.

By the definition of max we have $\max\{a, b\} \geq a$ and $\max\{a, b\} \geq b$ for any real numbers a and b. Therefore $\max\{x, -x\} \geq x$ and $\max\{x, -x\} \geq -x$. Using (i) we conclude $|x| \geq x$ and $|x| \geq -x$. This proves (iv). □

Exercise 2.3.2. Let $x \in \mathbb{R}$ and $a > 0$. Prove that $|x| < a$ if and only if $-a < x < a$.

Exercise 2.3.3. (a) Let $a, b \in \mathbb{R}$. Prove that $|a + b| \leq |a| + |b|$.

(b) Let $x, y, z \in \mathbb{R}$. Prove that $|x - y| \leq |x - z| + |z - y|$.

(c) Let $x, y \in \mathbb{R}$. Prove that $\big||x| - |y|\big| \leq |x - y|$.

Proof. Proof of (a). By Exercise 2.3.1 (iv) we know that $a \leq |a|$ and $b \leq |b|$. Therefore we conclude that

$$a + b \leq |a| + |b|. \tag{2.3.1}$$

By Exercise 2.3.1 (iv) we know that $-a \leq |a|$ and $-b \leq |b|$. Therefore we conclude

$$-(a + b) = -a + (-b) \leq |a| + |b|. \tag{2.3.2}$$

The inequalities (2.3.1) and (2.3.2) imply

$$\max\{a+b, -(a+b)\} \leq |a| + |b|. \tag{2.3.3}$$

By Exercise 2.3.1 (i) the inequality (2.3.3) yields $|a+b| \leq |a| + |b|$. This proves (a).
Prove (b) and (c) as an exercise. □

The inequalities in Exercise 2.3.3 are often called the *Triangle Inequalities.*

2.4 New functions from old

Definition 2.4.1. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subset \mathbb{R}$, and two real numbers α and β we form a new function $\alpha f + \beta g : A \to B$ defined by

$$(\alpha f + \beta g)(x) := a\, f(x) + \beta\, g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $\alpha\, f(x)$ and $\beta\, g(x)$ in the above formula is just a multiplication of real numbers. The function $\alpha f + \beta g$ is called a *linear combination* of the functions f and g.

Definition 2.4.2. Given two functions $f : A \to B$ and $g : A \to B$, with $A, B \subset \mathbb{R}$ we form a new function $fg : A \to B$ defined by

$$(fg)(x) := f(x)g(x), \quad \text{for all} \quad x \in A.$$

Notice that $f(x)$ and $g(x)$ are real numbers so that $f(x)g(x)$ in the above formula is just a multiplication of real numbers. The function fg is called the *product* of the functions f and g.

Definition 2.4.3. Given two functions $f : A \to B$ and $g : B \to C$ a new function $g \circ f : A \to C$ is defined by

$$(g \circ f)(x) := g(f(x)), \qquad x \in A.$$

The function $g \circ f$ is called the *composition* of the functions f and g.

Applying these definitions to familiar functions gives rise to new, sometimes very interesting functions.

2.5 More examples

Exercise 2.5.1. For each of the functions given below answer the following questions: (a) What are the domain and the range of the function? (b) Plot the function using your graphing calculator. Plot the function by hand emphasizing the details missed by your graphing calculator.

(a)	$x \mapsto x\,\mathrm{Abs}(x)$	(b)	$x \mapsto x(1 - \mathrm{Abs}(x))$
(c)	$x \mapsto x\,\mathrm{Sign}(x)$	(d)	$x \mapsto \mathrm{Ceiling}(x) - \mathrm{Floor}(x)$
(e)	$x \mapsto x - \mathrm{Floor}(x)$	(f)	$x \mapsto x\,\mathrm{Floor}(1/x)$
(g)	$x \mapsto (1 + \mathrm{Sign}(x))/2$	(h)	$x \mapsto x\ \mathrm{UnitStep}(x)$
(i)	$x \mapsto \mathrm{Sign}(\mathrm{Abs}(x))$	(j)	$x \mapsto \mathrm{Abs}(\mathrm{Sign}(x))$
(k)	$x \mapsto \mathrm{Floor}(\mathrm{Abs}(x))$	(l)	$x \mapsto \mathrm{Ceiling}(\mathrm{Abs}(x))$

3 Limit of a Function as x approaches $+\infty$

3.1 The definition

Definition 3.1.1. A function $x \mapsto f(x)$ has the limit L as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.

(II) For each real number $\epsilon > 0$ there exists a real number $X(\epsilon) \geq X_0$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

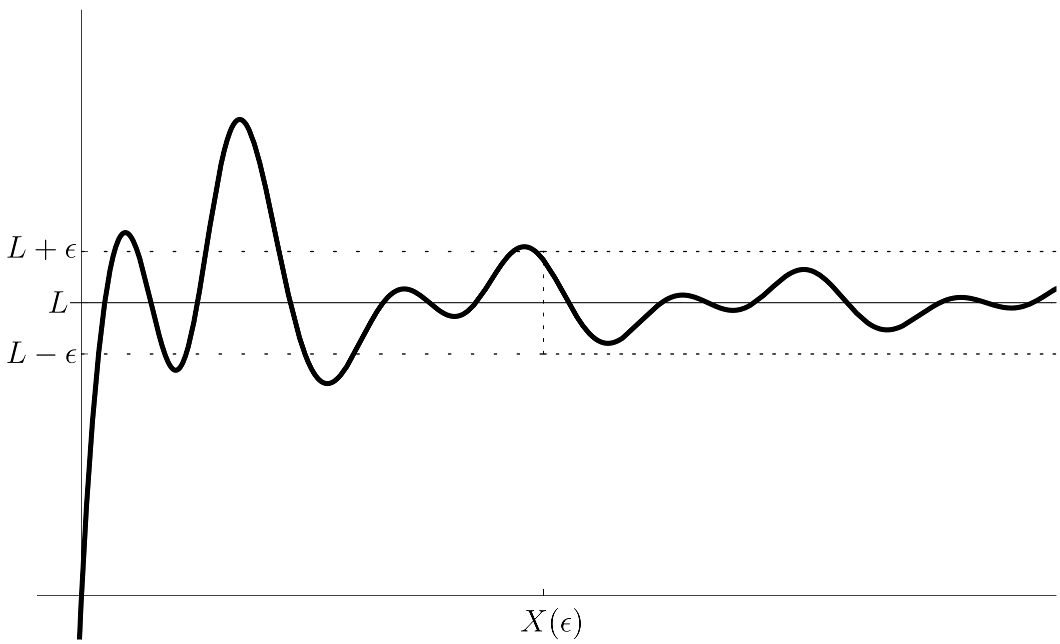

Figure 1: An illustration for (II) in Definition 3.1.1

If the conditions (I) and (II) in Definition 3.1.1 are satisfied we write $\lim\limits_{x\to+\infty} f(x) = L$.

3.2 Examples for Definition 3.1.1

Example 3.2.1. Prove that $\lim\limits_{x\to+\infty} \dfrac{1}{\sqrt{x-1}} = 0$.

Solution. We have to show that the conditions (I) and (II) in Definition 3.1.1 are satisfied. First we have to provide X_0. We can take $X_0 = 2$, since if $x \geq 2$, then $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 2$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{3.2.1}$$

In some sense we have to solve the inequality

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{3.2.2}$$

for x. The first step is to simplify it. Clearly

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| = \frac{1}{\sqrt{x-1}} \quad \text{for} \quad x \geq 2.$$

Thus we need to solve

$$\frac{1}{\sqrt{x-1}} < \epsilon. \tag{3.2.3}$$

This inequality is solved for x by using the following sequence of algebraic steps:

$$\frac{1}{\sqrt{x-1}} < \epsilon \quad \Leftrightarrow \quad \sqrt{x-1} > \frac{1}{\epsilon} \quad \Leftrightarrow \quad x - 1 > \frac{1}{\epsilon^2} \quad \Leftrightarrow \quad x > \frac{1}{\epsilon^2} + 1. \tag{3.2.4}$$

Since we need $X(\epsilon) \geq 2$, we choose $X(\epsilon) := \max\left\{ \frac{1}{\epsilon^2} + 1, 2 \right\}$.

It remains to prove that the implication (3.2.1) is satisfied. Assume that

$$x > X(\epsilon). \tag{3.2.5}$$

Since $X(\epsilon) \geq 2$, we conclude that $x > 2$. Therefore $x - 1 > 0$ and $1/\sqrt{x-1}$ is defined. Since $X(\epsilon) \geq 1/\epsilon^2 + 1$, we conclude that

$$x > \frac{1}{\epsilon^2} + 1. \tag{3.2.6}$$

Now the equivalences (3.2.4) imply that

$$\frac{1}{\sqrt{x-1}} < \epsilon. \tag{3.2.7}$$

Since $1/\sqrt{x-1}$ is positive we conclude that

$$\frac{1}{\sqrt{x-1}} = \left| \frac{1}{\sqrt{x-1}} \right| = \left| \frac{1}{\sqrt{x-1}} - 0 \right|. \tag{3.2.8}$$

Combining (3.2.7) and (3.2.8), yields

$$\left| \frac{1}{\sqrt{x-1}} - 0 \right| < \epsilon. \tag{3.2.9}$$

Thus, we have proved that the assumption (3.2.5) implies the inequality (3.2.9). This is exactly the implication (3.2.1). $\square$

Example 3.2.2. Determine the limit of the function $x \mapsto \dfrac{\text{Ceiling}(x)}{x}$ as x approaches $+\infty$ and prove your claim.

Solution. In Example 2.2.4 it is established that $x \leq \text{Ceiling}(x) < x + 1$ for each real number x. Therefore, for large x, the value of $\text{Ceiling}(x)$ does not differ much from x. Therefore it is reasonable to make the following claim

$$\lim_{x \to +\infty} \frac{\text{Ceiling}(x)}{x} = 1. \tag{3.2.10}$$

Next we shall prove this claim using Definition 3.1.1. Since the function $x \mapsto \dfrac{\text{Ceiling}(x)}{x}$ is defined for all $x \neq 0$, we can take $X_0 = 1$.

Next we show that the condition (II) is satisfied. Let $\epsilon > 0$ be given. We have to find a real number $X(\epsilon) \geq 1$ such that

$$x > X(\epsilon) \quad \Rightarrow \quad \left| \frac{\text{Ceiling}(x)}{x} - 1 \right| < \epsilon. \tag{3.2.11}$$

Solving for x the inequality

$$\left| \frac{\text{Ceiling}(x)}{x} - 1 \right| < \epsilon \tag{3.2.12}$$

is not easy. To find solutions of this inequality we first need to simplify it. In this process of simplification we can replace the expression

$$\left| \frac{\text{Ceiling}(x)}{x} - 1 \right|$$

with an expression which is greater or equal to it. In Example 2.2.4 we learned that

$$x \leq \text{Ceiling}(x) < x + 1. \tag{3.2.13}$$

Since we consider only $x \geq 1$, we can divide by x in (3.2.13) and subtract 1 from each term to get

$$0 \leq \frac{\text{Ceiling}(x)}{x} - 1 < \frac{x+1}{x} - 1 = \frac{1}{x}.$$

Therefore

$$\left| \frac{\text{Ceiling}(x)}{x} - 1 \right| \leq \frac{1}{x} \qquad \text{for all} \qquad x \geq 1. \tag{3.2.14}$$

This inequality is the key step in this proof. Therefore I call it the BIg INequality, or BiIn. (Each of the proofs involving the definition of limit involves a BIN.) The importance of BIN lies in the fact that instead of solving (3.2.12), we can solve for x the simpler inequality

$$\frac{1}{x} < \epsilon.$$

The solution of this inequality (remember $x \geq 1$) is $x > \dfrac{1}{\epsilon}$.

Now we can define $X(\epsilon) := \max\left\{\frac{1}{\epsilon}, 1\right\}$. With this $X(\epsilon)$ the implication (3.2.11) is true. It is easy to prove this claim: Assume that

$$x > X(\epsilon) = \max\left\{\frac{1}{\epsilon}, 1\right\}.$$

Then $x \geq 1$ and $x > \frac{1}{\epsilon}$. Since $x \geq 1$ the BIN inequality (see (3.2.14))

$$\left|\frac{\text{Ceiling}(x)}{x} - 1\right| \leq \frac{1}{x}$$

is true. Since also $x > \frac{1}{\epsilon}$, we conclude that

$$\frac{1}{x} < \epsilon.$$

The last two displayed inequalities imply that

$$\left|\frac{\text{Ceiling}(x)}{x} - 1\right| < \epsilon.$$

This proves the implication (3.2.11). $\square$

Exercise 3.2.3. Determine whether the following functions have limits as x approaches $+\infty$. Prove your statements using the definition.

(a) $x \mapsto \dfrac{x}{3x-2}$ (b) $x \mapsto \dfrac{2x}{x^2+x+1}$ (c) $x \mapsto \dfrac{x+\sin(x)}{x-1}$

(d) $x \mapsto \dfrac{x^2+x}{x^3+3}$ (e) $x \mapsto \dfrac{x^3-2x^2+1}{x^3+x+101}$ (f) $x \mapsto \sqrt{x+1}-\sqrt{x}$

(g) $x \mapsto \dfrac{x^2+x\cos(x)}{x^2-x+5}$ (h) $x \mapsto \left(\dfrac{1}{x}\right)^{1/\ln x}$ (i) $x \mapsto \dfrac{x^2-1}{x^2+2x\sin(x)}$

(j) $x \mapsto x - \sqrt{x^2-x}$

Exercise 3.2.4. Guess the limit of the function $x \mapsto \ln\left(1+\frac{1}{x}\right)^x$ and prove your guess.

Hint: 1) Use the rules for logarithms to simplify the expression. 2) Use the representation of the logarithm function $u \mapsto \ln(u)$ as an integral (area under the graph of the function $u \mapsto 1/u$) to find an upper and lower bound for the given function $x \mapsto \ln\left(1+\frac{1}{x}\right)^x$ for large values of x. The bounds should be very simple functions of x.

3.3 Negative results

How to prove that $\lim_{x\to+\infty} f(x) = L$ is false? This means that the condition (I) or the condition (II) in Definition 3.1.1 is not satisfied.

Next we formulate the negation of the condition (I): (In class I will explain how to formulate negations of statements involving "for all" and "there exists")

The negation of (I): For each $X \in \mathbb{R}$ there exists $x \geq X$ such that $f(x)$ is not defined.

Example 3.3.1. Prove that the function $f(x) = \dfrac{1}{x\,\mathrm{Sign}\big(\sin(x)\big)}$ does not satisfy the condition (I).

Solution. For this function the negation of (I) is true. This function is not defined for all $x = k\,\pi$ where $k \in \mathbb{Z}$. To prove that the negation of (I) is true let $X \in \mathbb{R}$ be arbitrary. Then

$$\pi\,\mathrm{Ceiling}\big(X/\pi\big) \geq X$$

and $f(x)$ is not defined for $x = \pi\,\mathrm{Ceiling}\big(X/\pi\big)$. $\square$

Below is the plot of the function f. Small circles indicate that this function is not defined at $x = \pi, 2\pi, 3\pi, \ldots, 9\pi$.

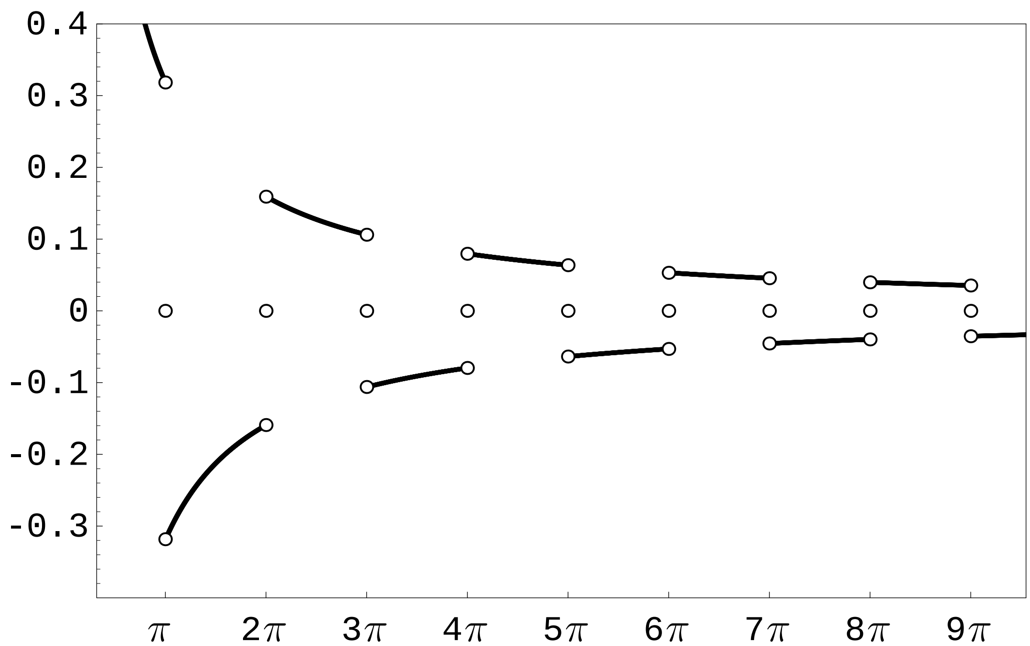

Figure 2: This function does not satisfy (I) in Definition 3.1.1

The negation of the condition (II) is more complicated:

The negation of (II): There exists $\epsilon > 0$ such that for every $X \in \mathbb{R}$ there exists $x > X$ such that $|f(x) - L| \geq \epsilon$.

Example 3.3.2. Prove that $\lim\limits_{x\to+\infty} \sin(x) = 0$ is false.

Solution. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$\pi \operatorname{Ceiling}\bigl(X/\pi\bigr) + \pi/2 > X$$

and, for $x = \pi \operatorname{Ceiling}\bigl(X/\pi\bigr) + \pi/2$, we have $|\sin(x)| = 1$. Therefore

$$|\sin(x) - 0| \geq 1/2. \qquad \square$$

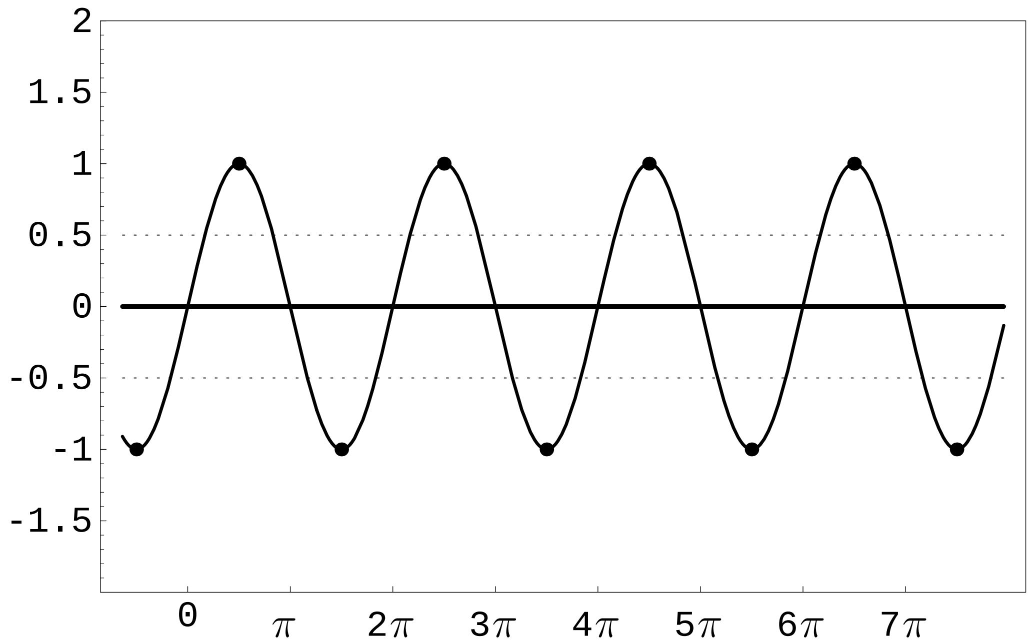

Figure 3: Illustration for the solution of Example 3.3.2

Now we consider the statement $\boxed{\text{“}\lim\limits_{x\to+\infty} f(x) \quad \text{does not exist.”}}$

This means that for each $L \in \mathbb{R}$, $\lim\limits_{x\to+\infty} f(x) = L$ is false.

Example 3.3.3. Prove that $\lim\limits_{x\to+\infty} \sin(x)$ does not exist.

Solution. Let $L \in \mathbb{R}$ be arbitrary. We need to prove that $\lim\limits_{x\to+\infty} \sin(x) = L$ is false. Consider two cases $L < 0$ and $L > 0$. Assume $L > 0$. Let $\epsilon = 1/2$. For arbitrary $X \in \mathbb{R}$ we have

$$2\pi \operatorname{Ceiling}\left(\frac{X}{2\pi}\right) + \frac{\pi}{2} > X$$

and, for $x = 2\pi \operatorname{Ceiling}\left(\frac{X}{2\pi}\right) + \frac{\pi}{2}$, we have $\sin(x) = 1$. Therefore

$$|\sin(x) - L| = |1 - L| = 1 + |L| \geq 1/2.$$

Do the case $L > 0$ as an exercise. $\square$

3.4 Infinite limits

Definition 3.4.1. A function $x \mapsto f(x)$ has the limit $+\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.

(II) For each real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) > M.$$

In this case we write $\lim\limits_{x\to+\infty} f(x) = +\infty$.

Definition 3.4.2. A function $x \mapsto f(x)$ has the limit $-\infty$ as x approaches $+\infty$ if the following two conditions are satisfied:

(I) There exists a real number X_0 such that $f(x)$ is defined for each $x \geq X_0$.

(II) For each real number M there exists a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad f(x) < M.$$

3.5 Examples of infinite limits

Example 3.5.1. Let $f(x) = \sqrt{x}$. Prove that $\lim\limits_{x\to+\infty} \sqrt{x} = +\infty$.

Solution. The function $\sqrt{\cdot}$ is defined for all $x \geq 0$. Therefore I can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. I have to determine a real number $X(M)$ such that

$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M.$$

This will be accomplished if I solve the inequality $\sqrt{x} > M$. If $M < 0$, then all $x \geq 0$ satisfy this inequality. If $M \geq 0$ then the solution of the inequality is $x > M^2$. Thus, I can take

$$X(M) = \begin{cases} M^2 & \text{if} \quad M \geq 0, \\ 0 & \text{if} \quad M < 0\,. \end{cases}$$

Clearly, $X(M) \geq 0$ for all $M \in \mathbb{R}$ and

$$x > X(M) \quad \Rightarrow \quad \sqrt{x} > M. \qquad \square$$

Example 3.5.2. Let $f(x) = \text{Floor}(x)$. Prove that $\lim\limits_{x\to+\infty} \text{Floor}(x) = +\infty$.

Solution. The function Floor is defined for all $x \in \mathbb{R}$. Therefore I can take $X_0 = 0$ in the part (I) of the definition.

Now consider the part (II) of the definition. Let $M \in \mathbb{R}$ be arbitrary. I have to determine a real number $X(M) \geq X_0$ such that

$$x > X(M) \quad \Rightarrow \quad \operatorname{Floor}(x) > M. \tag{3.5.1}$$

This will be accomplished if I solve the inequality

$$\operatorname{Floor}(x) > M. \tag{3.5.2}$$

Since we don't know much about Floor it is not easy to solve (3.5.2). To achieve the implication (3.5.1), we can replace $\operatorname{Floor}(x)$ in (3.5.2) with a smaller quantity $g(x)$ such that $g(x) > M$ is easy to solve. Thus we need $g(x)$ such that

(A) $\operatorname{Floor}(x) \geq g(x)$ for all $x > X_0$.

(B) $g(x) > M$ is easy to solve.

By the definition of $\operatorname{Floor}(x)$ we conclude that $0 \leq x - \operatorname{Floor}(x) < 1$ for all $x \in \mathbb{R}$. Therefore

$$x - 1 < \operatorname{Floor}(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.5.3}$$

Clearly $x - 1 > M$ is easy to solve: $x > M + 1$. Thus, I can take $X(M) = \max\{M + 1, 0\}$ in the part (II) of the definition. Clearly $X(M) \geq X_0 = 0$. Let $x > X(M)$. Then $x > M + 1$ and therefore $x - 1 > M$. By the inequality (3.5.3) I conclude that

$$\operatorname{Floor}(x) > x - 1 > M.$$

Thus $x > X(M)$ implies $\operatorname{Floor}(x) > M$. □

The key step in the solution of Example 3.5.2 was the discovery of the function $g(x)$ such that

(A) $f(x) \geq g(x)$ for all $x > X_0$.

(B) $g(x) > M$ is easy to solve.

Most proofs about limits follow this same pattern. I will sometimes refer to a discovery of the function g as a *Big Inequality*.

Exercise 3.5.3. Determine whether the following functions have the limit $+\infty$ when x approaches $+\infty$.

(a) $x \mapsto \dfrac{x^2}{2x+1}$ (b) $x \mapsto \ln x$ (c) $x \mapsto x - \sqrt{x}$

(d) $x \mapsto x - \ln(x)$ (e) $x \mapsto \dfrac{x^2 - x - 1}{x + 2\sqrt{x} + 1}$ (f) $x \mapsto \dfrac{1}{\sin\left(\frac{1}{x}\right)}$

(g) $x \mapsto \sqrt{x - \sqrt{x - \sqrt{x}}}$ (h) $x \mapsto \dfrac{(\cos x)^2 x}{\sqrt{x} + \sin(x)}$ (j) $x \mapsto \dfrac{(2 + \cos(x))x}{\sqrt{x} + \sin(x)}$